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## ON BIALOSTOCKI'S CONJECTURE FOR ZERO-SUM SEQUENCES

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**ABSTRACT.** Let  $n$  be a positive even integer, and let  $a_1, \dots, a_n$  and  $w_1, \dots, w_n$  be integers satisfying  $\sum_{k=1}^n a_k \equiv \sum_{k=1}^n w_k \equiv 0 \pmod{n}$ . A conjecture of Bialostocki states that there is a permutation  $\sigma$  on  $\{1, \dots, n\}$  such that  $\sum_{k=1}^n w_k a_{\sigma(k)} \equiv 0 \pmod{n}$ . In this paper we confirm the conjecture when  $w_1, \dots, w_n$  form an arithmetic progression with even common difference.

### 1. INTRODUCTION

A finite sequence  $S$  of terms from an (additive) abelian group is said to have zero-sum if the sum of the terms of  $S$  is zero. In 1961 P. Erdős, A. Ginzburg and A. Ziv [3] proved that any sequence of  $2n - 1$  terms from an abelian group of order  $n$  contains an  $n$ -term zero-sum subsequence. This celebrated EGZ theorem is an important result in combinatorial number theory and it has many different generalizations [5, 6, 7, 8] including Sun's recent extension involving covering systems.

The following theorem is called the weighted EGZ theorem. It was conjectured by Y. Caro [2] and proved by D. J. Grykiewicz [4].

**Theorem 1.1** (Weighted EGZ Theorem). *Let  $n$  be a positive integer and let  $w_1, \dots, w_n \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  with  $\sum_{k=1}^n w_k = 0$ . If  $a_1, a_2, \dots, a_{2n-1}$  is a sequence of elements from  $\mathbb{Z}_n$ , then  $\sum_{k=1}^n w_k a_{j_k} = 0$  for some distinct  $j_1, \dots, j_n \in \{1, \dots, 2n - 1\}$ .*

Recently Bialostocki raised the following challenging conjecture.

**Conjecture 1.1** (Bialostocki [1, Conjecture 14]). *Let  $n$  be a positive even integer. Suppose that  $a_1, \dots, a_n$  and  $w_1, \dots, w_n$  are zero-sum sequences with terms from  $\mathbb{Z}_n$ . Then there exists a permutation  $\sigma \in S_n$  such that  $\sum_{k=1}^n w_k a_{\sigma(k)} = 0$ , where  $S_n$  denotes the symmetric group of all permutations on  $\{1, \dots, n\}$ .*

The conjecture has been verified for  $n = 2, 4, 6, 8$ . It fails for  $n = 3, 5, 7, \dots$ . For example,  $\{a_1, a_2, a_3\} = \{w_1, w_2, w_3\} = \mathbb{Z}_3$  gives a counter-example for  $n = 3$ .

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In this paper we mainly establish the following result.

**Theorem 1.2.** *Let  $n$  be a positive even integer, and let  $a_1, \dots, a_n \in \mathbb{Z}$  with  $\sum_{k=1}^n a_k \equiv 0 \pmod{n}$ . Then there exists a permutation  $\sigma \in S_n$  such that  $\sum_{k=1}^n k a_{\sigma(k)} \equiv 0 \pmod{n/2}$ . Consequently, if  $w_1, \dots, w_n \in \mathbb{Z}$  form an arithmetic progression with even common difference, then  $\sum_{k=1}^n w_k a_{\sigma(k)} \equiv 0 \pmod{n}$  for some  $\sigma \in S_n$ .*

We are going to present two lemmas in the next section and then give our proof of Theorem 1.2 in Section 3.

## 2. TWO LEMMAS

**Lemma 2.1.** *Let  $n = mq$  with  $m, q \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$  and  $m \geq 2$ . Let  $d \in \mathbb{Z}^+$  be a divisor of  $q$ , and let  $a_1, \dots, a_n \in \mathbb{Z}$ . Then, there is a partition  $I_1, \dots, I_m$  of  $[1, n] = \{1, \dots, n\}$  such that for each  $s = 1, \dots, m$  we have  $|I_s| = q$  and*

$$d \mid \sum_{i \in I_s} a_i \implies |\{a_i \bmod d : i \in I_s\}| = 1.$$

*Proof.* By induction on  $m$ , it suffices to show that there exists an  $I \subseteq [1, n]$  with  $|I| = q$  such that for each  $J \in \{I, [1, n] \setminus I\}$  we have  $|\{a_j \bmod d : j \in J\}| = 1$  or  $\sum_{j \in J} a_j \not\equiv 0 \pmod{d}$ . To achieve this we distinguish three cases.

*Case 1.*  $|\{a_i \bmod d : i \in [1, n]\}| = 1$ .

In this case,  $I = [1, q]$  works for our purpose.

*Case 2.*  $|\{a_i \bmod d : i \in [1, n]\}| = 2$ .

Suppose that

$$\{a_i \bmod d : i \in [1, n]\} = \{r \bmod d, r' \bmod d\},$$

where  $r, r' \in [0, d-1]$ ,  $r \not\equiv r' \pmod{d}$ , and  $a_i \equiv r \pmod{d}$  for at least  $n/2$  values of  $i \in [1, n]$ . Choose  $I_0 \subseteq \{i \in [1, n] : a_i \equiv r \pmod{d}\}$  with  $|I_0| = q \leq n/2$ . Let  $i_0 \in I_0$  and  $j_0 \in \bar{I}_0 = [1, n] \setminus I_0$  with  $a_{j_0} \equiv r' \pmod{d}$ . When  $\sum_{j \in \bar{I}_0} a_j \equiv 0 \pmod{d}$ , we have both

$$\sum_{i \in (I_0 \setminus \{i_0\}) \cup \{j_0\}} a_i \equiv 0 - r + r' \not\equiv 0 \pmod{d}$$

and

$$\sum_{j \in (\bar{I}_0 \setminus \{j_0\}) \cup \{i_0\}} a_j \equiv 0 - r' + r \not\equiv 0 \pmod{d}.$$

Thus, there always exists an  $I \subseteq [1, n]$  with  $|I| = q$  such that

$$|\{a_i \bmod d : i \in I\}| = 1 \text{ or } \sum_{i \in I} a_i \not\equiv 0 \pmod{d},$$

and also  $\sum_{j \in \bar{I}} a_j \not\equiv 0 \pmod{d}$ .

*Case 3.*  $|\{a_i \bmod d : i \in [1, n]\}| > 2$ .

As  $n \geq 2q \geq 2q - 1$ , by the EGZ theorem there is an  $I_0 \subseteq [1, n]$  with  $|I_0| = q$  such that  $\sum_{i \in I_0} a_i \equiv 0 \pmod{q}$ . For  $\bar{I}_0 = [1, n] \setminus I_0$ , we clearly have  $|\bar{I}_0| = (m - 1)q$ . Set  $b = a_1 + \dots + a_n \equiv \sum_{j \in \bar{I}_0} a_j \pmod{q}$ .

Suppose that  $a_j - a_i \equiv 0$  or  $b \pmod{d}$  for any  $i \in I_0$  and  $j \in \bar{I}_0$ . Then

$$|\{a_i \bmod d : i \in I_0\}| \leq 2 \quad \text{and} \quad |\{a_j \bmod d : j \in \bar{I}_0\}| \leq 2.$$

If  $i_1, i_2 \in I_0$ ,  $j \in \bar{I}_0$  and  $a_j \not\equiv a_{i_1}, a_{i_2} \pmod{p}$ , then  $a_j - a_{i_1} \equiv b \equiv a_j - a_{i_2} \pmod{d}$  and hence  $a_{i_1} \equiv a_{i_2} \pmod{d}$ . So, if  $|\{a_i \bmod d : i \in I_0\}| = 2$  then  $\{a_j \bmod d : i \in \bar{I}_0\} \subseteq \{a_i \bmod d : i \in I_0\}$  which contradicts  $|\{a_i \bmod d : i \in I_0\}| > 2$ . Similarly, if  $|\{a_j \bmod d : j \in \bar{I}_0\}| = 2$  then we also have a contradiction. When

$$|\{a_i \bmod d : i \in I_0\}| = |\{a_j \bmod d : j \in \bar{I}_0\}| = 1,$$

we cannot have  $|\{a_i \bmod d : i \in [1, n]\}| > 2$ .

By the above, there are  $i_0 \in I_0$  and  $j_0 \in \bar{I}_0$  such that

$$a_{j_0} - a_{i_0} \not\equiv 0, b \pmod{d}.$$

Set

$$I = (I_0 \setminus \{i_0\}) \cup \{j_0\} \quad \text{and} \quad \bar{I} = [1, n] \setminus I = (\bar{I}_0 \setminus \{j_0\}) \cup \{i_0\}.$$

Then

$$\sum_{i \in I} a_i = \sum_{i \in I_0} a_i - a_{i_0} + a_{j_0} = 0 - a_{i_0} + a_{j_0} \not\equiv 0 \pmod{d}$$

and

$$\sum_{j \in \bar{I}} a_j = \sum_{j \in \bar{I}_0} a_j - a_{j_0} + a_{i_0} \equiv b + a_{i_0} - a_{j_0} \not\equiv 0 \pmod{d}.$$

Note that  $|I| = q$  and  $|\bar{I}| = (m - 1)q$ .

Combining the above and using the induction argument, we see that the desired result holds for any  $m = 2, 3, 4, \dots$   $\square$

**Lemma 2.2.** *Let  $a_1, \dots, a_n \in \mathbb{Z}$  with  $n = p^\alpha$ , where  $p$  is an odd prime and  $\alpha$  is a positive integer. If  $\sum_{k=1}^n a_k \not\equiv 0 \pmod{p}$  or  $|\{a_k \bmod p : k \in [1, n]\}| = 1$ , then there exists a permutation  $\sigma \in S_n$  such that  $\sum_{k=1}^n k a_{\sigma(k)} \equiv 0 \pmod{n}$ .*

*Proof.* If  $a := \sum_{k=1}^n a_k \not\equiv 0 \pmod{p}$ , then there is an  $l \in [1, n]$  such that  $al + \sum_{k=1}^n k a_k \equiv 0 \pmod{p}$  and hence

$$\sum_{k=1}^n k a_{\sigma(k)} \equiv \sum_{k=1}^n (k + l) a_k \equiv \sum_{k=1}^n k a_k + l a \equiv 0 \pmod{p^\alpha},$$

where  $\sigma(k)$  is the least positive residue of  $k - l$  modulo  $n$ .

In the case  $a_1 \equiv \cdots \equiv a_n \pmod{p}$ , it is clear that

$$\sum_{k=1}^p ka_k \equiv a_1 \sum_{k=1}^p k = a_1 p \frac{p+1}{2} \equiv 0 \pmod{p}.$$

Thus we have the desired result for  $\alpha = 1$ .

Now let  $\alpha > 1$  and assume the desired result with  $\alpha$  replaced by  $\alpha - 1$ . As mentioned above, the desired result holds if  $\sum_{k=1}^n a_k \not\equiv 0 \pmod{p}$ . Suppose that  $a_1 \equiv \cdots \equiv a_n \pmod{p}$  and set  $b_k = (a_k - a_1)/p$  for  $k = 1, \dots, n$ . In light of Lemma 2.1, there exists a partition  $I_1 \cup \cdots \cup I_p$  of  $[1, n]$  with  $|I_1| = \cdots = |I_p| = p^{\alpha-1}$  such that for any  $s = 1, \dots, p$  either  $|\{b_k \pmod{p} : k \in I_s\}| = 1$  or  $\sum_{k \in I_s} b_k \not\equiv 0 \pmod{p}$ . By the induction hypothesis, there are one-to-one mappings  $\sigma_s : [1, p^{\alpha-1}] \rightarrow I_s$  ( $s = 1, \dots, p$ ) such that

$$\sum_{k=1}^{p^{\alpha-1}} kb_{\sigma_s(k)} \equiv 0 \pmod{p^{\alpha-1}} \quad \text{for all } s = 1, \dots, p.$$

For  $s \in [1, p]$  and  $t \in [1, p^{\alpha-1}]$  define  $\sigma(p^{\alpha-1}(s-1) + t) = \sigma_s(t)$ . Then  $\sigma \in S_n$  and

$$\begin{aligned} \sum_{k=1}^n ka_{\sigma(k)} &= \sum_{k=1}^n ka_1 + p \sum_{k=1}^k kb_{\sigma(k)} \\ &= \frac{p^\alpha(p^\alpha + 1)}{2} a_1 + p \sum_{s=1}^p \sum_{t=1}^{p^{\alpha-1}} (p^{\alpha-1}(s-1) + t) b_{\sigma_s(t)} \\ &\equiv p \sum_{s=1}^p \sum_{t=1}^{p^{\alpha-1}} tb_{\sigma_s(t)} \equiv 0 \pmod{p^\alpha}. \end{aligned}$$

This concludes the induction step and we are done.  $\square$

### 3. PROOF OF THEOREM 1.2

*Proof of Theorem 1.2.* We use induction on  $\nu(n)$ , the total number of prime divisors of  $n$ .

In the case  $\nu(n) = 1$ , clearly  $n = 2$  and the desired result holds trivially.

Now let  $\nu(n) > 1$  and assume the desired result for those even positive integers with less than  $\nu(n)$  prime divisors.

*Case 1.*  $n = 2^\alpha$  for some  $\alpha \geq 2$ .

By the EGZ theorem, there is an  $I \subseteq [1, n]$  with  $|I| = n/2 = 2^{\alpha-1}$  such that  $\sum_{i \in I} a_i \equiv 0 \pmod{2^{\alpha-1}}$ . Note that for  $\bar{I} = [1, n] \setminus I$  we also have

$$\sum_{j \in \bar{I}} a_j = \sum_{k=1}^n a_k - \sum_{i \in I} a_i \equiv 0 \pmod{2^{\alpha-1}}.$$

By the induction hypothesis, for some one-to-one mappings  $\sigma_0 : [1, n/2] \rightarrow I$  and  $\sigma_1 : [1, n/2] \rightarrow \bar{I}$  we have

$$2 \sum_{k=1}^{2^{\alpha-1}} k a_{\sigma_0(k)} \equiv 2 \sum_{k=1}^{2^{\alpha-1}} k a_{\sigma_1(k)} \equiv 0 \pmod{2^{\alpha-1}}.$$

Observe that

$$\sum_{k=1}^{2^{\alpha-1}} (2k-1) a_{\sigma_1(k)} \equiv 2 \sum_{k=1}^{2^{\alpha-1}} k a_{\sigma_1(k)} - \sum_{j \in \bar{I}} a_j \equiv 0 \pmod{2^{\alpha-1}}.$$

For  $k \in [1, n/2]$  and  $r \in [0, 1]$  define  $\sigma(2k-r) = \sigma_r(k)$ . Then  $\sigma \in S_n$  and

$$\sum_{j=1}^n j a_{\sigma(j)} = 2 \sum_{k=1}^{2^{\alpha-1}} k a_{\sigma_0(k)} + \sum_{k=1}^{2^{\alpha-1}} (2k-1) a_{\sigma_1(k)} \equiv 0 \pmod{2^{\alpha-1}}.$$

Thus we have the desired result for  $n = 2^\alpha$ .

*Case 2.*  $n$  has an odd prime divisor  $p$ .

Write  $n = p^\alpha m$  with  $\alpha, m > 0$  and  $p \nmid m$ . With the help of Lemma 2.1 there is a partition  $I_1 \cup \dots \cup I_m$  of  $[1, n]$  with  $|I_1| = \dots = |I_m| = p^\alpha$  such that for each  $s = 1, \dots, m$  either  $|\{a_i \bmod p : i \in I_s\}| = 1$  or  $\sum_{i \in I_s} a_i \not\equiv 0 \pmod{p}$ . Combining this with Lemma 2.2, we see that for each  $s \in [1, m]$  there is a one-to-one mapping  $\sigma_s : [1, p^\alpha] \rightarrow I_s$  such that  $\sum_{t=1}^{p^\alpha} t a_{\sigma_s(t)} \equiv 0 \pmod{p^\alpha}$ .

Set  $b_s = \sum_{k \in I_s} a_k$  for  $s = 1, \dots, m$ . Then

$$\sum_{s=1}^m b_s = \sum_{k \in I_1 \cup \dots \cup I_m} a_k = \sum_{k=1}^n a_k \equiv 0 \pmod{m}.$$

As  $2 \mid m$  and  $\nu(m) < \nu(n)$ , by the induction hypothesis, for some  $\tau \in S_m$  we have

$$2 \sum_{s=1}^m s b_{\tau(s)} \equiv 0 \pmod{m}$$

and hence

$$2 \sum_{s=1}^m \sum_{t=1}^{p^\alpha} s a_{\sigma_{\tau(s)}(t)} = 2 \sum_{s=1}^m s b_{\tau(s)} \equiv 0 \pmod{m}.$$

Note also that

$$\sum_{s=1}^m \sum_{t=1}^{p^\alpha} t a_{\sigma_{\tau(s)}(t)} = \sum_{s=1}^m \sum_{t=1}^{p^\alpha} t a_{\sigma_s(t)} \equiv 0 \pmod{p^\alpha}.$$

Therefore

$$2 \sum_{s=1}^m \sum_{t=1}^{p^\alpha} (p^\alpha s + mt) a_{\sigma_{\tau(s)}(t)} \equiv 0 \pmod{p^\alpha m}.$$

As  $p^\alpha$  is relatively prime to  $m$ ,

$$\{p^\alpha s + mt : s \in [1, m] \text{ and } t \in [1, p^\alpha]\}$$

is a complete system of residues modulo  $n = p^\alpha m$ . For any  $k \in [1, n]$ , there are unique  $s \in [1, m]$  and  $t \in [1, p^\alpha]$  such that  $k \equiv p^\alpha s + mt \pmod{n}$ , and we define  $\sigma(k) = \sigma_{\tau(s)}(t)$ . Then  $\sigma \in S_n$  and also

$$2 \sum_{k=1}^n k a_{\sigma(k)} \equiv 0 \pmod{n}.$$

This concludes the induction step.

In view of the above, we have completed the proof of Theorem 1.2.  $\square$

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